SERIES

Σ

A mathematical ***series*** is a fancy term for **sum**, or summation. A **finite** series might be the summation

$$1 + 2 + 5 + 6 + 10 \qquad \text{(which has a } \textbf{sum} \text{ of 24)}$$

An **infinite** series might be something like this:

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\cdots$$

Believe it or not, this *infinite series* actually has a sum of $\frac{1}{2}$.

where the three dots (called an *ellipsis*) mean to go on and on forever (infinitely many terms being added).

Sometimes, if there's a nice pattern, a series can be written in "**sigma**" notation. The Greek capital letter sigma, Σ, is used to represent the summation (since sigma and sum both begin with the letter ***s***). The following examples should explain how this notation should be read and calculated.

<u>EXAMPLE 1:</u> **Evaluate:** $\displaystyle\sum_{k=2}^{5}(2k+1)$

<u>Solution:</u> The expression that determines the numbers we will add together is $2k + 1$. Below the sigma sign is the <u>starting</u> value of the index variable k, in this case 2; above the sigma sign is the <u>ending</u> value of k, in this case 5. So k starts at 2 and ends at 5, but what does k do in between? We agree that it goes up by <u>one</u> — in other words, k will go 2, 3, 4, and then 5. Check it out, remembering that the sigma sign, Σ, means ADD:

$$\sum_{k=2}^{5}(2k+1) = \underset{(k=2)}{(2\cdot\mathbf{2}+1)} + \underset{(k=3)}{(2\cdot\mathbf{3}+1)} + \underset{(k=4)}{(2\cdot\mathbf{4}+1)} + \underset{(k=5)}{(2\cdot\mathbf{5}+1)}$$

$$= 5 + 7 + 9 + 11$$

$$= \boxed{\mathbf{32}}$$

EXAMPLE 2: **Evaluate:** $\displaystyle\sum_{n=0}^{3}(n^2 - n)$

Solution: Does it matter that Example 1 used the index variable k and this example uses the index variable n? Not at all — the variable used makes no difference in the final answer; it's just a placeholder. The values of n will be 0, 1, 2, and 3. For each value of n we evaluate the expression $n^2 - n$. Then we add up the results.

$$\sum_{n=0}^{3}(n^2 - n) = \underset{(n=0)}{(\mathbf{0}^2 - \mathbf{0})} + \underset{(n=1)}{(\mathbf{1}^2 - \mathbf{1})} + \underset{(n=2)}{(\mathbf{2}^2 - \mathbf{2})} + \underset{(n=3)}{(\mathbf{3}^2 - \mathbf{3})}$$

$$= 0 + 0 + 2 + 6$$

$$= \boxed{\mathbf{8}}$$

Homework

Evaluate each series (calculate each sum):

1. $\displaystyle\sum_{k=3}^{5}(7k-1)$

2. $\displaystyle\sum_{n=2}^{5}(n^2+n)$

3. $\displaystyle\sum_{k=1}^{6}\frac{1}{2^k}$

4. $\displaystyle\sum_{j=0}^{4}3^j$

5. $\displaystyle\sum_{n=16}^{16}\frac{1}{2}\sqrt{n}$

6. $\displaystyle\sum_{k=3}^{5}\frac{1}{k}$

7. $\sum_{m=0}^{2} \frac{1}{m+1}$

8. $\sum_{t=-1}^{4} 2t$

9. $\sum_{j=-2}^{2} j^3$

For an extra challenge, write each summation in Σ-notation; use *k* as the index variable, and start at *k* = 1:

10. $10+11+12+13$

11. $4^3 + 5^3 + 6^3$

12. $1+\sqrt{2}+\sqrt{3}+2$

13. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

14. $1+\sqrt[3]{2}$

15. $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$

Review Problems

16. $\sum_{n=2}^{5} (n^2+n) =$

17. $\sum_{k=1}^{6} \frac{1}{2^k} =$

18. $\sum_{j=0}^{4} 3^j =$

19. $\sum_{n=16}^{16} \frac{1}{2}\sqrt{n} =$

20. $\sum_{k=1}^{10} k =$

21. $\sum_{n=0}^{4} 2^{n-2} =$

22. True/False:

a. $\sum_{i=2}^{5} i^2 = 54$

b. $\sum_{j=0}^{3} 2^j = 16$

c. $\sum_{k=-1}^{1} k^3 = 3$

d. $\sum_{L=0}^{4} \sqrt{L} = 3+\sqrt{2}+\sqrt{3}$

23. [Only if you've studied *Combinations*]:

$$\sum_{k=0}^{4} \binom{4}{k} a^{4-k} b^k \ =$$

This can also be written $\sum_{k=0}^{4} C(4, k)\, a^{4-k}\, b^k$.

❑ *To ∞ and Beyond*

A. Evaluate: $\sum_{k=1}^{99} \left(\frac{1}{k} - \frac{1}{k+1} \right)$

B. Evaluate: $\sum_{k=1}^{\infty} \frac{1}{2^k}$

C. Evaluate: $\sum_{k=1}^{\infty} \frac{1}{k}$

[Research]

D. Evaluate: $\sum_{n=1}^{4} \frac{(-1)^{n+1}}{n^3}$

[A decimal answer will be fine.]

E. Write in Σ-notation. Use k for the index variable, starting with $k = 0$.

$$\frac{1}{2} + \frac{3}{3} + \frac{5}{6} + \frac{7}{11} + \frac{9}{18} + \frac{11}{27}$$

Solutions

1. 81

2. 68

3. $\frac{63}{64}$

4. 121

5. 2

6. $\frac{47}{60}$

7. $\frac{11}{6}$

8. 18

9. 0

10. $\sum_{k=1}^{4} (k+9)$

11. $\sum_{k=1}^{3} (k+3)^3$

12. $\sum_{k=1}^{4} \sqrt{k}$

13. $\sum_{k=1}^{5} \frac{1}{k}$

14. $\sum_{k=1}^{2} \sqrt[3]{k}$

15. $\sum_{k=1}^{4} \frac{1}{2^{k+1}}$

16. 68

17. $\frac{63}{64}$

18. 121

19. 2

20. 55

21. $\frac{31}{4}$

22. a. T b. F c. F d. T

23. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, which can also be written $(a + b)^4$

Check out: **Math is Fun - Infinite Series**
https://www.mathsisfun.com/algebra/infinite-series.html

“The highest activity a human being can attain is learning for understanding, because to understand is to be free.”

– Baruch Spinoza